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Thermodynamics of a dusty plasma

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In the present work, we develop the thermodynamics of a dusty plasma and give an equation of state for two cases: (a) when the dust forms a stationary background and the charge on the grain does not vary and (b) when the dust charge fluctuates either due to the fluctuations in the electron and ion number densities or due to the dust density variation. Application of the results to the various space plasma situations has been indicated.

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I. INTRODUCTION

The dynamics of dusty plasmas have been a subject of intense investigation in the last decade. The dusty plasma is ubiquitous in the space environment. For example, planetary rings, cometary tails, and the interstellar medium are typical sites of dusty plasmas [1,2]; the closest examples of naturally occurring dusty plasmas have been noctilucent clouds (at altitudes of 80–90 km) and magnetospheric dust [1]. Owing to its large inertia, dust grains bring new spatial and temporal scales to the plasma dynamics that is manifested in low-frequency oscillations such as dust-acoustic and dust-ion-acoustic waves. The low-frequency collective behavior of a dusty plasma does not greatly affect the collective behavior of electrons and ions, since over the dust time scale, plasma particles reach statistical equilibrium. The fluctuation of the charge on the grain introduces novel collective features to the dusty plasma. In the absence of charge dynamics, dusty plasmas are more like a rescaled version of the multicomponent plasma. However, the fluctuation on the grain charge introduces a "fast" time scale and the collective behavior of the electrons and ions could be altered due to fluctuation of the grain charge [3].

The charge on the dust grain can fluctuate due to grain collision with plasma particles, due to the photoionization and secondary emission, and due to density fluctuations of the dust grain. A self-consistent charging model has been utilized recently [4] to study the effect of dust grains on ion-acoustic waves [5]. The charge on the grain can also fluctuate due to the orbit-limited plasma current fluctuation when the dust grain collects plasma particles at a random interval.

The ideal gas equation can be applied with sufficient accuracy in many cases. However, for a real gas, when molecular interaction is important, there is a considerable departure from the perfect gas equation, and one must account for such a departure. In fact, the strength of the molecular interaction is responsible for the phase transition in any thermodynamic system [6]. Furthermore, charged particles bring an additional complication to this picture due to the novel collective behavior of a charged system—i.e., due to fluctuation of the charge on dust grains. In the present work, we derive an equation of state for a dusty plasma system.

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II. DERIVATIONS

We shall consider an unmagnetized dusty plasma consisting of electrons, singly charged ions, and equal-radius spherical grains carrying identical charges. The dusty plasma as a whole should be electrically neutral and thus

$$e(n_{i0}-n_{e0})+q_d n_{d0}=0, \tag{1}$$

where $n_{e0,i0,d0}$ are the number densities of electrons, ions, and dust grains and $q_d=|Z_d|e$ is the positive/negative charge on the dust grain.

The charged dust grain in the plasma creates around itself an inhomogeneous charged cloud (on average spherically symmetrical [6]) of opposite polarity which shields its electric field. The potential energy of the dust-cloud system is $|q_d|\varphi$, where φ is the potential of the dust-cloud system. We shall assume that the electron and ion number density is given by the Boltzmann distribution

$$n_e=n_{e0}\exp\left(\frac{e\varphi}{T_e}\right), \tag{2}$$

$$n_i=n_{i0}\exp\left(\frac{-e\varphi}{T_i}\right), \tag{3}$$

where $n_{e0,i0}$ are the number densities far away from the charged cloud. Assuming that the dust particles form only a uniform fixed background due to their great inertia, one may write Poisson's equation as

$$\varepsilon_0\nabla^2\varphi=(en_e-en_i-q_d n_d), \tag{4}$$

i.e., the potential φ of the field in the dust cloud is related to the charge density in it. Generally, it is assumed that the behavior of a dusty plasma deviates only slightly from an ideal gas. This can be valid only if the electrostatic energy of plasma particles in the cloud, caused by the collective interaction, is much less than the average kinetic energy—i.e., $|e|\varphi\ll T_{e,i}$. Then, after solving Poisson's equation (4) by retaining terms only up to linear in φ in n_e and n_i, in the background of fixed dust particles, one obtains the Debye-Hückel potential

$$\varphi_{\mathrm{DH}}=\frac{1}{4\pi\varepsilon_0}\frac{q_d}{r}\exp\left(-\frac{r}{\lambda_D}\right), \tag{5}$$

where $\lambda_D^{-2}=\lambda_{De}^{-2}+\lambda_{Di}^{-2}\equiv n_{e0}e^2/\varepsilon_0 T_e+n_{i0}e^2/\varepsilon_0 T_i$. The ideal gas assumption ($|e|\varphi \ll T_{e,i}$) reduces the nonlinear Poisson equation (4) to a linear equation in φ. However, often, the charge on the grain is large. Therefore, the expansion $n_{e,i}(\varphi)$ in terms of φ [Eqs. (2) and (3)] should include at least a term up to the second order—i.e., $n_{e,i}(\varphi)=A\varphi+B\varphi^2+\cdots$. The solution of the resulting nonlinear Poisson equation (4) is [7]

$$\hat{\varphi}(r)=\frac{C}{r/\lambda_D}\exp\left(-\frac{r}{\lambda_D}\right)-\frac{C^2}{4}\frac{\exp\left(\frac{r}{\lambda_D}\right)}{r/\lambda_D}\mathrm{Ei}\left(-\frac{3r}{\lambda_D}\right)+\frac{C^2}{4}\frac{\exp\left(-\frac{r}{\lambda_D}\right)}{r/\lambda_D}\mathrm{Ei}\left(-\frac{r}{\lambda_D}\right). \qquad (6)$$

Here $\hat{\varphi}\equiv e\varphi/T$ is the normalized potential and $\mathrm{Ei}(-r)$ is the exponential integral. In the immediate neighborhood of dust, the field must become the Coulomb field of the given charge q_d and thus, $C=q_d e/4\pi\varepsilon_0 T\lambda_D$. The Debye length λ_D may be regarded as determining the size of the dust cloud produced by the given dust. In a dusty plasma, the Debye length is larger than the interdust distance. In order to obtain the cloud potential—i.e., $\varphi_{\mathrm{cloud}}=\lim[\varphi_{\mathrm{DH}}-\varphi_{\mathrm{Coulomb}}]$—we expand the above potential as a series and find that, for small r/λ_D,

$$\varphi_{\mathrm{cloud}}=-Z_d g_e f-\frac{4}{3}Z_d^2 g_e^2 f^2. \qquad (7)$$

Here

$$g_e=\frac{1}{4\pi n_{e0}\lambda_{De}^3} \qquad (8)$$

is the correlation parameter and is an indicator of the deviation of a dusty plasma from an ideal gas and $f=(2+Z_d n_{d0}/n_{e0})^{-0.5}$. Equation (7) has been derived by Taylor expansion of Eq. (6) in the $\epsilon\equiv r/\lambda_D\to 0$ limit. In the $\epsilon\to 0$ limit, the first term of Eq. (6) [after neglecting $O(\epsilon^2)$ and higher-order terms] gives the Coulomb potential and potential due to the Debye cloud. Similary, expansion of the second and third terms gives a correction to the Coulomb potential and Debye cloud potential, respectively. One notes that use has been made of the leading term of the exponential integral—i.e., $\mathrm{Ei}(-r)\approx -\exp[-1/r]/r$—in deriving Eq. (7).

To calculate the thermodynamic quantities of a dusty plasma gas, one must first determine the increase in the Coulomb energy E_{Coul} in comparison with the ideal gas, due to the interaction between the plasma particles. The electrostatic energy of a system of charged dust grains in the field of all other particles is

$$E_{\mathrm{Coul}}=\frac{V}{2}\Sigma q_d n_{d0}\varphi_{\mathrm{cloud}}. \qquad (9)$$

Here V is the total volume of the gas. Making use of the cloud potential, Eq. (7), one can write the Coulomb energy as

$$E_{\mathrm{Coul}}=-\frac{Z_d^2 T}{2\sqrt{V}}\Sigma G_e N_{d0}F'\left[1+\frac{4Z_d}{3\sqrt{V}}G_e F'\right]. \qquad (10)$$

Here, $N_{\alpha 0}=Vn_{\alpha 0}$ is the total number of particles, $T_e=T_i=T$, $\lambda_{De}=\sqrt{\varepsilon_0 T/N_e e^2}$, $G_e=1/4\pi N_{e0}\lambda_{De}^3$, and $F'=(2+Z_d N_{d0}/N_{e0})^{-0.5}$. It is reasonable to assume that the electrons and ions are at the same temperature in the interstellar medium [8]. However, in the interplanetary medium, electron and ion temperatures may not be equal to each other. For example, observational evidence suggests that the ion temperature may be an order of magnitude larger than the electron temperature in the Earth's tail [9]. Thus our assumption of equal plasma temperature will underestimate the Coulomb energy in interplanetary situations. Making use of the thermodynamic relation $E/T^2=-[\partial/\partial T(F/T)]$, we can obtain the free energy corresponding to the Coulomb energy E_{Coul} as

$$F=F_p-\frac{Z_d^2 T}{3\sqrt{V}}\Sigma G_e N_{d0}F'\left[1+\frac{2Z_d}{3\sqrt{V}}G_e F'\right]. \qquad (11)$$

Here F_p is the free energy of a perfect gas. Using $P=-\partial F/\partial V$, we can calculate the equation of state for a dusty plasma system:

$$PV=T\Sigma N_{d0}-\frac{Z_d^2 T}{6\sqrt{V}}\Sigma G_e N_{d0}F'\left[1+\frac{4Z_d}{3\sqrt{V}}G_e F'\right]. \qquad (12)$$

The departure from an ideal gas behavior [the terms in the square bracket in Eq. (12)] is proportional to Z_d^2 and Z_d^3. The number of charges Z_d on the grain may vary from 10^3 to 10^4 in the interplanetary environment [10]. In the interstellar medium, grain charge may vary between 1 and 10. For example, in the H II region, the typical charge on the grain around a circumstellar cloud is about 10 [11]. As a result, the departure from an ideal gas behavior in a dusty plasma can be very strong. One notes that the term proportional to Z_d^3 in Eq. (12) is solely due to the correction in the Debye-Hückel potential [the last two terms in Eq. (6)]. It has been pointed out in [7] that the nonlinear correction may cause the potential to change sign at some critical ϵ. This circumstance leads to a modification of the equation of state for a dusty plasma. In order to see the role of the linear versus nonlinear terms on the equation of state, we plot $PV/T\Sigma N_{d0}$ (y axis) against the size of the electronic charge Z_d (x axis) on the dust grain for correlation parameter $g=0.001$ and $Z_d n_{d0}/n_{e0}=1$ and $N_{d0}=100$. In Fig. 1, curve B represents Eq. (12) in the absence of the last term and curve A is plotted when all the terms in Eq. (12) are retained. Clearly, the departure from the ideal gas behavior is severe when proper account of the dust charge field is taken into consideration.

The thermodynamic potential for a dusty plasma can be easily calculated and is given as

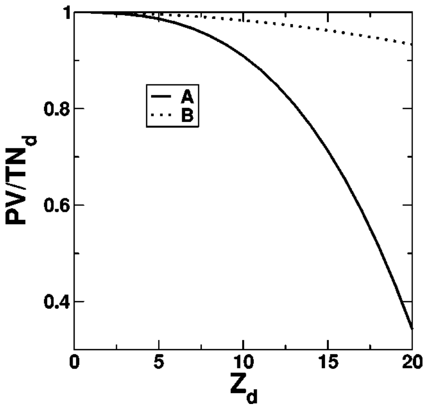

FIG. 1. Equation of state for a constant grain charge, Eq. (12), for $g=0.001$, $Z_d n_{d0}/n_{e0}=1$, and $N_{d0}=100$. Curve B corresponds to a situation when the nonlinear effect of the grain charge on the potential [last term, Eq. (12)] is unimportant. Curve A retains the effect of the nonlinear term as well.

$$\Phi=\Phi_p-\frac{Z_d^2}{6}\sqrt{\frac{PT}{\Sigma N_{d0}}}\Sigma G_e N_{d0}F'\left[1+\frac{2Z_d}{3\Sigma N_{d0}}G_e F'\right]. \tag{13}$$

The generalization of the above equation of state, for a charge fluctuation model based on orbit-limited motion (OLM) theory, is straightforward. In the presence of charge fluctuations, the Poisson equation for stationary grains becomes

$$\varepsilon_0\nabla^2\varphi=(en_e-en_i-q_{d0}n_d-q_{d1}n_d), \tag{14}$$

where the dust number density has been assumed constant and q_{d1} is the dust charge fluctuation. Making use of the OLM electron and ion currents and imposing $I_e+I_i=0$, one can write dust charge fluctuation q_{d1} (where $q_{d1}\ll q_{d0}$) as [12]

$$q_{d1}=-\frac{\nu_1}{\nu_2}a\varphi, \tag{15}$$

where

$$\nu_1=\frac{a}{\sqrt{2\pi}}\left[\frac{\omega_{pe}}{\lambda_{De}}\exp\left(\frac{eq_{d0}}{aT_e}\right)+\frac{\omega_{pi}}{\lambda_{Di}}\left(1-\frac{eq_{d0}}{aT_e}\right)\right] \tag{16}$$

and

$$\nu_2=\frac{a}{\sqrt{2\pi}}\left[\frac{\omega_{pe}}{\lambda_{De}}\exp\left(\frac{eq_{d0}}{aT_e}\right)+\frac{\omega_{pi}}{\lambda_{Di}}\right]. \tag{17}$$

Here, a is the grain radius and $\omega_{p\alpha}=(n_\alpha e^2/\varepsilon_0 m_\alpha)^{1/2}$ is the plasma frequency of the αth plasma species. In deriving Eqs. (16) and (17) it has been assumed that the electrons and ions behave like a perfect gas and fluctuation in the grain potential energy is much less than the plasma random energy. Making use of Eq. (15), the Poisson equation (14) may be recast as Eq. (6) replacing λ_D^{-2} with an effective Debye length λ_c^{-2}, where λ_c is

$$\lambda_c=\frac{\lambda_{Di}}{\sqrt{1+\left(\frac{\lambda_{Di}}{\lambda_{De}}\right)^2+\left(\frac{\lambda_{Dg}}{\lambda_{Di}}\right)^2}}, \tag{18}$$

with $\lambda_{Dg}^{-2}=\varepsilon_0\nu_2/n_{d0}a\nu_1$. For $T_e=T_i=T$ and $\nu_2=\nu_1$, with $\lambda_{De}=\sqrt{f}\lambda_{Di}$, one may write

$$\lambda_c=\frac{\lambda_{De}}{\sqrt{1+f^2+f^4\left(\frac{\lambda_{Dg}}{D_{De}}\right)^2}}. \tag{19}$$

The expression for the cloud potential is

$$\varphi_{\text{cloud}}=-Z_d g_e\sqrt{1+f^2+f^4\left(\frac{\lambda_{Dg}}{\lambda_{De}}\right)^2}-\frac{4}{3}Z_d^2g_e^2\left[1+f^2+f^4\left(\frac{\lambda_{Dg}}{\lambda_{De}}\right)^2\right]. \tag{20}$$

In deriving Eq. (20) from Eq. (6), we have retained only up to $\sim O(\epsilon)$ terms in the expansion. Clearly, then, Eq. (20) is valid in the $\epsilon\to 0$ limit only. The $\epsilon\to 0$ limit captures the effect of the Coulomb field as well as the effect of the modification to the Coulomb field by the Debye screening. Defining $R=c+b/T$, where $c=1+f^2$ and $b=f^4\,(N_{e0}/N_{d0})\,(e^2/a)$, one may write the Coulomb energy as

$$E_{\text{Coul}}=-\frac{Z_d^2T}{2\sqrt{V}}\Sigma N_{d0}G_e\sqrt{R}\left[1+\frac{4Z_d}{3\sqrt{V}}G_e\sqrt{R}\right]. \tag{21}$$

Then the free energy of the system can be written as

$$F=F_p-\frac{Z_d^2T}{4\sqrt{V}}\Sigma N_{d0}G_e\left\{\frac{R}{b/T}\left[1-\frac{1}{2R}+\frac{1}{2\sqrt{b/T}R^{1.5}}\ln\left|\frac{\sqrt{b+cT}-\sqrt{b}}{\sqrt{b+cT}+\sqrt{b}}\right|\right]+\frac{8Z_d}{9\sqrt{V}}G_e\left(c+\frac{3b}{4T}\right)\right\}. \tag{22}$$

From the above equation, one can easily calculate the equation of state for a charge-fluctuating dusty plasma, which is

$$PV=T\Sigma N_{d0}-\frac{Z_d^2T}{8\sqrt{V}}\Sigma N_{d0}G_e\left\{\frac{R}{b/T}\left[1-\frac{1}{2R}+\frac{1}{2\sqrt{b/T}R^{1.5}}\ln\left|\frac{\sqrt{b+cT}-\sqrt{b}}{\sqrt{b+cT}+\sqrt{b}}\right|\right]+\frac{16Z_d}{9\sqrt{V}}G_e,\left(c+\frac{3b}{4T}\right)\right\}. \tag{23}$$

Finally, we shall consider the variation in the dust density. One can anticipate on physical grounds that the charge on the dust must be related to the number density, which in turn will be related to the dust surface potential and the plasma potential φ [13]. The dependence of the plasma potential on the dust density gives rise to an electric force [14]

$$\vec{F}_e = -q_d \nabla \varphi = -q_d(n_d) \frac{\partial \varphi}{\partial n_d} \nabla n_d . \tag{24}$$

As has been noted in [14], this force is zero for a uniform density, a situation considered above. However, when the dust density fluctuates—i.e., $n_d = n_{d0} + n_{d1}$—then

$$q_d \nabla \varphi = q_{d0} \frac{\partial \varphi_0}{\partial n_{d0}} \nabla n_{d0} + \nabla \left(q_{d0} \frac{\partial \varphi_0}{\partial n_{d0}} n_{d1} \right). \tag{25}$$

The last term acts like a pressure with an effective temperature $T_{\rm eff} = q_{d0} \partial \varphi_0 / \partial n_{d0}$. This "electrostatic pressure" can excite acousticlike modes with a phase velocity $v_{\rm ph} = T_{\rm eff}/m_d)^{1/2}$. It is obvious that such a mode will operate on the "slow" dust time scale and, therefore, the equation of state of a dusty plasma may be affected by such fluctuations in the system. In order to be able to consider the charge fluctuation due to density fluctuations, we can solve Eq. (4) by replacing λ_D^{-2} by λ_c^{-2}, where [15]

$$\lambda_c^{-2} = d \tau \lambda_{De}^{-2}. \tag{26}$$

Here, $d = Z_d n_{d0}/n_{e0}$, $\tau = T/T_{\rm eff}$.

The cloud potential is

$$\varphi_{\rm cloud} = -Z_d g_e \tau^{0.5} d^{0.5} - \frac{4}{3} Z_d^2 g_e^2 \tau d. \tag{27}$$

The Coulomb energy of the cloud potential can be written as

$$E_{\rm Coul} = -\frac{Z_d^2 T}{2\sqrt{V}} \tau^{0.5} \Sigma N_{d0} d^{0.5} G_e \left(1 + \frac{4 Z_d}{3\sqrt{V}} G_e \tau^{0.5} d^{0.5} \right). \tag{28}$$

The free energy is given as

$$F = F_p - \frac{Z_d^2 T}{2\sqrt{V}} \tau^{0.5} \Sigma N_{d0} d^{0.5} G_e \left(1 + \frac{2 Z_d}{3\sqrt{V}} G_e \tau^{0.5} d^{0.5} \right). \tag{29}$$

The corresponding equation of state becomes

$$PV = T \Sigma N_{d0} - \frac{Z_d^2 T}{4\sqrt{V}} \tau^{0.5} \Sigma N_{d0} d^{0.5} G_e \left(1 + \frac{4 Z_d}{3\sqrt{V}} G_e \tau^{0.5} d^{0.5} \right). \tag{30}$$

The thermodynamic potential can be similarly constructed.

III. SUMMARY

To summarize, the thermodynamic quantities for a dusty plasma have been derived for two cases: (a) when the dust charge fluctuation is unimportant and (b) when the dust charge fluctuates either due to the fluctuation of the plasma particles or due to the fluctuation in the dust number density. The departure from the ideal gas behavior is proportional to the Z_d^2 and Z_d^3. In the interplanetary environment, a grain may carry between 10 and 10^4 electronic charges [10]. Therefore, in a interplanetary environment, dusty plasma is a strongly nonideal gas. In the H I and H II regions as well, the thermodynamic state of a dusty plasma is a nonideal gas, since $Z_d \sim 10-100$ [11]. Thus one may conclude that in the space and astrophysical environment, dusty plasma is a nonideal gas and the phase transition in such a gas (viz., crystal formation, etc.) should be described using the equations of state given in the present work.

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